THE FUNDAMENTAL CONTINUITY THEORY OF OPTIMIZATION ON A COMPACT SPACE I

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Abstract: Optimizers on a compact feasible region are abstractly specified and, as set-valued mappings, are studied for sufficient conditions yielding them (as well as certain associated maps and certain restrictions of all these) continuous, using function space methods. In particular, the study concerns the continuity of the set of optimal solutions as a function of the three arguments: (i) objective function used, (ii) an incentive (or "penalty/reward") function imposed, and (iii) an abstract "parameter". An interpretation of the mathematical apparatus is suggested and a brief game-theoretic illustration given.

Thanks are due Miss Janet Allen for kindly (and meticulously) typing what follows.

Notation: Let S and T be topological spaces. The set of all continuous mappings of S into T will be denoted by T^S. Given $B \subset S$ and $W \subset T$, the set $\{f \in T^S \mid f(B) \subset W\}$ will be denoted by (B, W). $\mathcal{P}(S)$ will denote the power set of S and $\mathcal{C}(S)$ will denote the set of non-empty closed subsets of S. Given $B \subset S$, $\langle B \rangle$ will denote the set $\{A \in \mathcal{C}(S) \mid A \subset B\}$. Throughout, R will be a non-empty set equipped with a total order $\geq$ and a (Hausdorff) topology which is at least as fine as the order topology.[0]

Aim: Given non-empty topological spaces $X = X_1 \times X_2$ and Y, of which X is compact with X_2 (compact and) Hausdorff, in this paper we seek sufficient conditions for the continuity of (a) the optimizers

$$\alpha'\colon \; R^{X\times Y} \times Y^X \times X_2 \to \mathcal{P}(X_1)$$

defined by the specification

$$\alpha'(u,g,x_2) = \{x_1 \in X_1 \mid u(x_1,x_2,g(x_1,x_2)) \geq \sup_{X_1} u(\cdot,x_2,g(\cdot,x_2))\}$$
$$(u \in R^{X\times Y},\; g \in Y^X,\; x_2 \in X_2)$$

[a specification which is quickly seen to yield α' as being into $\mathcal{C}(X_1)$], or (b) of its "associate"

$$\alpha\colon \; R^{X\times Y} \times Y^X \to \mathcal{P}(X_1)^{X_2}$$

determined by $\alpha(u, g)(x_2) = \alpha'(u, g, x_2)$, or (c) of the restrictions

$$\alpha'_{(u,g)} = \alpha'|\{(u,g)\}\times X_2 : \to \mathcal{P}(X_1) \text{ of } \alpha' \text{ to } \{(u,g)\} \times X_2,$$

$$\alpha'_u = \alpha'|_{\{u\}\times Y^X \times X_2}: \to \mathcal{P}(X_1) \text{ of } \alpha' \text{ to } \{u\} \times Y^X \times X_2,$$

$$\alpha_u = \alpha|_{\{u\}\times Y^X}: \to \mathcal{P}(X_1)^{X_2} \text{ of } \alpha \text{ to } \{u\} \times Y^X.$$

Motivation: Our motivation derives from a desire to understand in what sense optimization on a compact space (X_1 above) is continuous, a question we consider to be perhaps the fundamental topological question of the theory of constrained optimization.

Interpretation: The functions $u \in R^{X\times Y}$ may be called utility functions or objective functions, perhaps the former being preferred so as to emphasize their merely ordinal nature. These are understood to be representations of complete (i.e., "total" or "decisive") transitive relations on $X \times Y$. Concerning the existence of such continuous representations we refer the reader to [Debreu, 1954] and [Sertel, 1971b].

For the functions $g \in Y^X$ we prefer the term incentive function [Sertel, 1971a] to the more restrictive terms "payoff function", "penalty/reward function", etc. (although we use the term "payoff function" in a game-theoretic illustration below).

For R we refer back to our Notation[0]. As to X_1, it plays the role of the familiar feasible region. The role of Y is that the utility functions u are in general sensitive, not only to $x \in X$, but also to $y \in Y$, so that externally instituted rules $g \in Y^X$ may serve to influence optimal behavior, i.e., $\alpha'(u, g, x_2)$. X_2 may be considered as an abstract "parameter space" in which a point is exogenously determined, subject to which optimization is to be carried out.

A game-theoretic context may serve to illustrate. For simplicity,

consider a non-cooperative two-person game in which X_1 and X_2 are the strategy spaces of players 1 and 2, respectively. Then g specifies the "payoff" to player 1, for whom u serves as generic utility function, and α_u determines the player (1) completely: for mathematical purposes, we are interested essentially in $\alpha_u(g)$: $X_2 \to \mathcal{P}(X_1)$, and would study how this ("reaction function") varies as g is varied, given the "tastes" or "preferences" as represented by u. Here we also investigate how, from the viewpoint of continuity,[1] the whole map α, as well as its associate α' and the restrictions listed under (c) above, behaves.

Method/Procedure: The nature of the investigation makes it natural to proceed in the reverse order $\langle$(c), (b), (a)$\rangle$ in studying the continuity of the maps listed above. We always take the upper semi-finite topology on $\mathcal{C}(X_1)$ and equip the function spaces $R^{X\times Y}$, Y^X and $\mathcal{C}(X_1)^{X_2}$ with their respective compact-open topologies. First we study $\alpha'_{(u,g)}$ and α_u (Theorem I). Then we turn to α'_u in the Corollary to Theorem I. Finally, we study α and α' in Theorem II, taking Y locally compact.

Topological Refresher: Let S and T be topological spaces. The compact-open topology on the function space T^S is the one generated by $\{(B, W) \mid B \subset S$ is compact and $W \subset T$ is open$\}$ as sub-base.

The upper semi-finite topology on $\mathcal{C}(T)$ is the one generated by $\{\langle W\rangle \mid W \subset T$ is open$\}$ as sub-base. A point-to-set map α: $S \to \mathcal{C}(T)$ is said to be upper semi-continuous[1] iff it is continuous when $\mathcal{C}(T)$ is given the upper semi-finite topology.[2] It is easily seen that α: $S \to \mathcal{C}(T)$ is upper semi-continuous iff the set $\{s\varepsilon S \mid \alpha(s) \cap D \neq \emptyset\}$ is closed for every closed $D \subset T$. [Michael, 1951].

<u>Proposition 1</u>: Let X, Y and Z be topological spaces, of which X is locally compact, and let $u \in Z^{(X\times Y)}$. Define a map ω_u: $Y^X \to Z^X$ by writing $\omega_u(g) = \hat{g}_u$ and $\hat{g}_u(x) = u(x, g(x))$ for every $g \in Y^X$ and $x \in X$. Then ω_u is continuous.

<u>Proof</u>: Giving Y^X and Z^X their respective compact-open topologies, let $(B, W) \subset Z^X$ be a subbasic open set, and take any $g \in Y^X$ such that $\omega_u(g) \in (B, W)$. Denote $G = \omega_u^{-1}((B, W))$ and $H = u^{-1}(W)$. Thus, $g \in G$ and, by continuity of u, $H \subset X \times Y$ is open. Hence, for every $b \in B$, $(b, g(b)) \in H$ and there exists an open box nbd $A_b \times V_b \subset H$ of $(b, g(b))$; furthermore, using continuity of g, we may assume $g(A_b) \subset V_b$; and, since X is locally compact, we may actually assume that A_b is relatively compact (i.e., its closure $\bar{A}_b$ is compact) with $g(\bar{A}_b) \subset V_b$ and $\bar{A}_b \times V_b \subset H$. Now $\{A_b \mid b \in B\}$ is an open cover of (compact) B, affording a finite subcover $\{A_{b_i} \mid i=1, \ldots, m\}$. Defining

$$A = \bigcap_{i=1}^{m} (\bar{A}_{b_i}, V_{b_i}),$$

we see that $A \subset Y^X$ is a basic open set with $g \in A \subset G$, showing that $G \subset Y^X$ is open and proving that ω_u is continuous.

#

<u>Lemma 1</u>: Let S and T be compact spaces, of which S is Hausdorff. Let γ: $S \to \mathcal{C}(T)$ be a map whose graph $\Gamma = \{(s, t) \in S \times T \mid t \in \gamma(s)\}$ is compact. Then γ is upper semi-continuous.

<u>Proof</u>: Let $D \subset T$ be closed. Then D is compact, since T is so. Since S is compact, $S \times D$ is also compact. Thus, $\Gamma \cap (S \times D)$ is compact, so that its projection P into S is compact. But P is precisely the set $\{s \in S \mid \gamma(s) \cap D \neq \emptyset\}$; and, being compact in the Hausdorff space S, P is closed. This proves that γ is upper semi-continuous.

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<u>Proposition 2</u>: Let $X = X_1 \times X_2$ be a non-empty compact space with X_2 Hausdorff; and, for every $v \in R^X$, define $\hat{\alpha}(v)\colon X_2 \to \mathcal{P}(X_1)$ by

$$\hat{\alpha}(v)(x_2) = \{x_1 \in X_1 \mid v(x_1, x_2) \geq \operatorname*{Sup}_{X_1} v(\cdot, x_2)\},$$

so that $v \mapsto \hat{\alpha}(v)$ determines a map $\hat{\alpha}$ on R^X. Then

(1) $\hat{\alpha}$ maps R^X into $\mathcal{C}(X_1)^{X_2}$ [i.e., for each $v \in R^X$, $\hat{\alpha}(v)$ maps X_2 upper semi-continuously into $\mathcal{C}(X_1)$]; and

(2) $\hat{\alpha}\colon R^X \to \mathcal{C}(X_1)^{X_2}$ is continuous.

<u>Proof</u>: (<u>ad</u> (1)): Let $v \in R^X$. For every $x_2 \in X_2$, $X_1 \times \{x_2\}$ is compact, so the continuous v attains a supremum on a non-empty closed subset, specifically on $\hat{\alpha}(v)(x_2) \times \{x_2\} \subset X_1 \times \{x_2\}$, whereby $\hat{\alpha}(v)(x_2) \in \mathcal{C}(X_1)$, and we see that $\hat{\alpha}(v)$ maps X_2 into $\mathcal{C}(X_1)$. To prove that $\hat{\alpha}(v)\colon X_2 \to \mathcal{C}(X_1)$ is upper semi-continuous, using Lemma 1, it suffices to show that the graph $\hat{\Gamma} = \{(x_2, x_1) \in X \mid x_1 \in \hat{\alpha}(v)(x_2)\}$ is compact, and this we now do. From the continuity of v and compactness of X_1, (it is straightforward to show that) the function $\bar{v}\colon X_2 \to R$ determined by $\bar{v}(x_2) = \operatorname*{Sup}_{X_1} v(\cdot, x_2)$ is well-defined and continuous. Since v and $\bar{v}$ are continuous and their range is Hausdorff, their graphs $\Gamma_v = \{(x, r) \in X \times R \mid r = v(x)\}$ and $\Gamma_{\bar{v}} = \{(x_2, r) \in X_2 \times R \mid r = \bar{v}(x_2)\}$, respectively, are closed. Also, by continuity of v and compactness of X, v(X) is compact. Thus, the closed set $K = (X_1 \times \Gamma_{\bar{v}}) \cap \Gamma_v \subset X \times v(X)$ is compact. Now the projection of K into X is precisely the graph $\hat{\Gamma}$, so $\hat{\Gamma}$ is compact and, by Lemma 1, (1) is proved.

(<u>ad</u> (2)): Agreeing on the notation $<\!|\, P, Q \,|\!> = \{v \in R^X \mid (p, q) \in P \times Q \Rightarrow v(p) < v(q)\}$ for subsets $P, Q \subset X$, first we note that $<\!|\, P, Q \,|\!> \subset R^X$ is open when P and Q are compact[3]: $<\!|\, P, Q \,|\!>$ is just the union $\left(\bigcup_{t \in R} ((P, (-\infty, t)) \cap (Q, (t, +\infty)))\right) \cup \left(\bigcup_{(\underline{s}, \bar{s}) \in \Omega} ((P, (-\infty, \bar{s})) \cap (Q, (\underline{s}, +\infty)))\right)$, where[3] $\Omega = \{(\underline{s}, \bar{s}) \in R \times R \mid \underline{s} < \bar{s}$ and there is no $t \in R$ with $\underline{s} < t < \bar{s}\}$.

Take any compact non-empty $B \subset X_2$ and any non-empty open $W \subset X_1$, so that $(B, \langle W \rangle) \subset \mathcal{C}(X_1)^{X_2}$ is a subbasic open set, and take any $v \in R^X$ such that $\hat{\alpha}(v) \in (B, \langle W \rangle)$. <u>Claim</u> (established in the next paragraph): for every $b \in B$, there is a relatively compact open nbd $V(b)$ of b and a compact set $K(b) \subset W$ such that $<\!|\, \bar{V}(b) \times W^c, \bar{V}(b) \times K(b) \,|\!>$, where $W^c = X_1 \setminus W$, is a nbd of v. In that case, $\{V(b) \mid b \in B\}$ is an open cover of B affording a finite subcover $\{V(b_i) \mid i = 1, \ldots, m\}$, and it follows that $G = \bigcap_{i=1}^{m} <\!|\, \bar{V}(b_i) \times W^c, \bar{V}(b_i) \times K(b_i) \,|\!>$ is a nbd of v such that $\hat{\alpha}(G) \subset (B, \langle W \rangle)$, showing that $\hat{\alpha}$ is continuous. To complete the proof, we turn to establish our claim above.

For each $b \in B$, v attains a supremum $\bar{s}(b) = \operatorname{Sup}_{\{b\} \times W} v$ on $\{b\} \times W$, and (since W^c is compact) $\underline{s}(b) = \operatorname{Sup}_{\{b\} \times W^c} v$ on $\{b\} \times W^c$. Now, for all $b \in B$, $\underline{s}(b) < \bar{s}(b)$ holds, and there are just two possibilities: (i) R owns an element $t(b)$ such that $\underline{s}(b) < t(b) < \bar{s}(b)$, and (ii) R owns no such element. If (i) holds, define $T_*(b) = (-\infty, t(b))$ and $T^*(b) = (t(b), +\infty)$. As these are both open sets and v continuous, taking any $w(b) \in W$ with $v(b, w(b)) \in T^*(b)$, using (local) compactness of X, we may find a relatively compact open box nbd $V_1(b) \times U(b)$ of $(b, w(b))$ such that (the closure) $\bar{V}_1(b) \times \bar{U}(b) \subset v^{-1}(T^*(b))$. As W^c is compact, we may also find a

relatively compact open nbd $V_2(b)$ of b such that $\bar{V}_2(b) \times W^c \subset v^{-1}(T_*(b))$. Thus, writing $V(b) = V_1(b) \cap V_2(b)$ and $K(b) = \bar{U}(b)$, $<|\ \bar{V}(b) \times W^c,\ \bar{V}(b) \times K(b)\ |>$ is a nbd of v. If (ii) holds, define $T_*(b) = (-\infty, \bar{s}(b))$ and $T^*(b) = (\underline{s}(b) +\infty)$, and find $V(b)$ and $K(b)$ as in the case of (i). This completes the proof.

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<u>FUNDAMENTAL CONTINUITY THEOREM I</u>: Let $X = X_1 \times X_2$ and Y be non-empty topological spaces, of which X is compact with X_2 Hausdorff, and let $u \in R^{(X\times Y)}$. Consider the function[4] α_u on Y^X which determines, for each $g \in Y^X$, a (unique) map[4] $\alpha_u(g)$ defined on X_2 by

$$\alpha_u(g)(x_2) = \{x_1 \in X_1 \mid u(x_1, x_2, g(x_1, x_2)) \geq \operatorname*{Sup}_{X_1} u(\cdot, x_2, g(\cdot, x_2))\}.$$

Then

(1) α_u maps Y^X into $\mathcal{C}(X_1)^{X_2}$ [i.e., for each $g \in Y^X$, $\alpha_u(g)$ maps X_2 upper semi-continuously into $\mathcal{C}(X_1)$]; and

(2) α_u is continuous.

<u>Proof</u>: (<u>ad</u> (1)): Let $g \in Y^X$. By continuity of u, the function $\hat{g}_u = \omega_u(g)$ defined in Proposition 1 is continuous, i.e., $\hat{g}_u \in R^X$. Thus, Proposition 2(1) applies, showing that $\alpha_u(g) = \hat{\alpha}(\hat{g}_u) \in \mathcal{C}(X_1)^{X_2}$, and (1) is proved.

(<u>ad</u> (2)): Proposition 2(2) shows that $\alpha_u(g) = \hat{\alpha}(\hat{g}_u)$ is continuous in $\hat{g}_u$, and from Proposition 1 we see that $\hat{g}_u$ is continuous in g ($g \in Y^X$). Thus, as the composition $\alpha_u = \hat{\alpha} \circ \omega_u$: $Y^X \to \mathcal{C}(X_1)^{X_2}$, α_u is continuous, as to be shown. This completes the proof.

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<u>COROLLARY</u>: In Theorem I, the map $\alpha'_u\colon Y^X \times X_2 \to \mathcal{C}(X_1)$ defined by

$$\alpha'_u(g, x_2) = \alpha_u(g)(x_2) \qquad\qquad (g \in Y^X,\ x_2 \in X_2)$$

is continuous (in both arguments <u>together</u>).

<u>Proof</u>: From Theorem I, we already see that α'_u is continuous in each argument ($g \in Y^X$ and $x_2 \in X_2$) separately. But X_2 is compact, hence locally compact, in which case the result is a well-known consequence[5] of the compact-open topology (on $\mathcal{C}(X_1)^{X_2}$).

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<u>Lemma 2</u>: Let X, Y and Z be non-empty topological spaces of which X and Y are locally compact.[6] Define the maps μ': $Z^{X\times Y} \times Y^X \times X \to Z$ and μ: $Z^{X\times Y} \times Y^X \to Z^X$ by $\mu'(u, g, x) = u(x, g(x))$ and $\mu(u, g)(\cdot) = \mu'(u, g, \cdot)$ $(u \in Z^{X\times Y}, g \in Y^X, x \in X)$. Then

(1) μ' is continuous; and

(2) μ, too, is continuous.

<u>Proof</u>: (<u>ad</u> (1)): Take any (u, g, x) in the domain of μ', and let $W \subset Z$ be any open nbd of $\mu'(u, g, x)$. Then $(x, g(x)) \in u^{-1}(W)$, so continuity of u yields an open box nbd $U \times V$ of $(x, g(x))$ with $U \subset X$ and $V \subset Y$ such that $u(U \times V) \subset W$. As X and Y are locally compact, we may assume $u(\bar{U} \times \bar{V}) \subset W$; and, since g is continuous, we may also assume that $g(\bar{U}) \subset V \subset \bar{V}$. Now we have $N = (\bar{U} \times \bar{V}, W) \times (\bar{U}, V) \times U$ a nbd of (u, g, x) with $\mu'(N) \subset W$, showing that μ' is continuous.

(<u>ad</u> (2)): This follows directly from (1) and the well-known fact that the compact-open topology on Z^X is always splitting.[7]

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<u>FUNDAMENTAL CONTINUITY THEOREM II:</u> Let $X = X_1 \times X_2$ be a non-empty compact space with X_2 Hausdorff, and let Y be a non-empty locally compact space.[6] Consider the map α' on $R^{X\times Y} \times Y^X \times X_2$ defined (with reference to Theorem I) by $\alpha'(u, g, x_2) = \alpha_u(g)(x_2)$, and consider the "associate" map α on $R^{X\times Y} \times Y^X$ determined by $\alpha(u, g) = \alpha_u(g)$ ($u \in R^{X\times Y}$, $g \in Y^X$, $x_2 \in X_2$). Then α' is into $\mathcal{C}(X_1)$ and α is into $\mathcal{C}(X_1)^{X_2}$, and both maps

$$\alpha': \; R^{X\times Y} \times Y^X \times X_2 \to \mathcal{C}(X_1)$$
$$\alpha: \; R^{X\times Y} \times Y^X \to \mathcal{C}(X_1)^{X_2}$$

are continuous.

<u>Proof</u>: In Lemma 2(2), set $Z = R$. Then $\mu: R^{X\times Y} \times Y^X \to R^X$ is continuous. Also, using Proposition 2, $\hat{\alpha}: R^X \to \mathcal{C}(X_1)^{X_2}$ is continuous. Now, clearly, $\alpha = \hat{\alpha} \circ \mu: R^{X\times Y} \times Y^X \to \mathcal{C}(X_1)^{X_2}$ is continuous and α' is into $\mathcal{C}(X_1)$. Since X_2 is compact, hence locally compact, the compact-open topology on $\mathcal{C}(X_1)^{X_2}$ is conjoining,[7] so α', too, is continuous. This completes the proof.

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Historical Note: An earlier study of the continuity of the set of optimal solutions is [Dantzig, Folkman and Shapiro, 1967]. It works with feasible regions lying in a metric space and with objective functions coinciding with real-valued incentive functions, and takes the "parameter space" X_2 as singleton, investigating the continuity of the set of optimal solutions as a function of (i) objective function used and (ii) feasible region. Thus, neither does it contain, nor is it contained in, the present study.

In a paper following the present and [Sertel, 1971a], we intend to study the continuity of the set of optimal solutions as a function of the feasible region as well. (The present study extends a corrected version of the author's "The Fundamental Continuity Theory of Optimization on a Compact Hausdorff Space", Alfred P. Sloan School of Management, Working Paper 620-72, October, 1972.)

FOOTNOTES

0. Our notation R is intended to be mnemonic of the real line, i.e., the set of all real numbers taken with the usual (Euclidean) topology, since objective functions in the literature are almost always specified as real-valued. (N.B. The usual topology on the set of reals coincides, of course, with the order topology, using the natural order of the reals). Thus, the reader may wish to interpret R, throughout this paper, as the real line; mathematically, (s)he is quite free to do so.

1. A terminological warning may be in order. For, according to a terminology not used here, a mapping into a space of non-empty closed sets is called "continuous" iff it is continuous when the range space is given its "finite" topology (see [Michael, 1951]), a topology finer than the upper semi-finite. When we are given a map of a topological space into a space of non-empty closed sets and the topology on the range is understood (e.g., to be the upper semi-finite), we feel free to speak of the map as continuous iff it is so w.r.t. the topology on the domain and that on the range!

2. Note that a function $\Psi^*: S \to \mathcal{C}(T)$ which is singleton-valued (i.e., such that, for each $s \in S$, $\Psi^*(s)$ is a one-element set) is upper semi-continuous iff the function $\Psi: S \to T$, defined by $\Psi^*(s) = \{\Psi(s)\}$ $(s \in S)$, is continuous.

3. Denoting $(R \times R) \setminus \geq$ by $<$, by $(-\infty, \bar{s})$ we mean $\{r \in R \mid r < \bar{s}\}$ and by $(\underline{s}, +\infty)$ we mean $\{r \in R \mid \underline{s} < r\}$.

4. Strictly speaking, as made clear in announcing our Aim, α_u is on $\{u\} \times Y^X$ and $\alpha_u(g) = \alpha'_{(u,g)}$. As $\{u\} \times Y^X$ is homeomorphic to Y^X, our informality in treating α_u as a map on Y^X should cause no error.

5. See, e.g., Theorem 3.1 (2) on pp. 261 of [Dugundji, 1966].

6. Notice that, if Y is the real line, it is locally compact. The relevance of this is that our theorem may be applied in the study of optimization subject to real-valued (e.g., monetary) incentive functions, such as in the usual economic case: e.g., the case of taxes or subsidies, the case of wages or salaries, and the general case of a price system.

7. See § 10 of Ch. XII in [Dugundji, 1966].

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